

University of Groningen

On k-step CSCS-based polynomial preconditioners for Toeplitz linear systems with application to fractional diffusion equations

Gu, Xian-Ming; Huang, Ting-Zhu; Li, Hou-Biao; Li, Liang; Luo, Wei-Hua

Published in:
Applied Mathematics Letters

DOI:
[10.1016/j.aml.2014.11.005](https://doi.org/10.1016/j.aml.2014.11.005)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2015

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Gu, X-M., Huang, T-Z., Li, H-B., Li, L., & Luo, W-H. (2015). On k-step CSCS-based polynomial preconditioners for Toeplitz linear systems with application to fractional diffusion equations. *Applied Mathematics Letters*, 42, 53–58. <https://doi.org/10.1016/j.aml.2014.11.005>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.



On k -step CSCS-based polynomial preconditioners for Toeplitz linear systems with application to fractional diffusion equations[☆]

Xian-Ming Gu^{a,b}, Ting-Zhu Huang^{a,*}, Hou-Biao Li^a, Liang Li^a, Wei-Hua Luo^a

^a School of Math. Sci., University of Electronic Science and Technology of China, Chengdu, 611731, China

^b Institute of Math. Comput. Sci., University of Groningen, Nijenborgh 9, P.O. Box 407, 9700 AK Groningen, The Netherlands

ARTICLE INFO

Article history:

Received 24 September 2014

Received in revised form 4 November 2014

Accepted 4 November 2014

Available online 22 November 2014

Keywords:

Fractional diffusion equation

CSCS iteration method

Toeplitz matrix

Polynomial preconditioner

Fast Fourier transform

ABSTRACT

The implicit finite difference scheme with the shifted Grünwald formula for discretizing the fractional diffusion equations (FDEs) often results in the ill-conditioned non-Hermitian Toeplitz systems. In the present paper, we consider to solve such Toeplitz systems by exploiting the preconditioned GMRES method. A k -step polynomial preconditioner is designed based on the circulant and skew-circulant splitting (CSCS) iteration method proposed by Ng (2003). Theoretical and experimental results involving numerical solutions of FDEs demonstrate that the proposed k -step preconditioner is efficient to accelerate the GMRES solver for non-Hermitian Toeplitz systems.

© 2014 Elsevier Ltd. All rights reserved.

1. Introduction

In this paper, we are interested in the iterative solvers for non-Hermitian Toeplitz systems arising in the numerical solutions of FDEs via finite difference method; see [1,2] and references therein. We also denote the Toeplitz system as the following form of matrices product

$$A\mathbf{u} = \mathbf{f}, \quad A \in \mathbb{C}^{n \times n} \text{ nonsingular, and } \mathbf{u}, \mathbf{f} \in \mathbb{C}^n, \quad (1)$$

where A is said to be Toeplitz if $A = [A_{ij}]_{i,j=1}^n = [a_{i-j}]$, $i \geq j$, i.e., A is constant along its diagonals; see [3,4]. Toeplitz systems also come from a variety of other applications in mathematics and engineering, see for instance the references in [3,4]. These applications have motivated both mathematicians and engineers to develop specific algorithms catering to solving Toeplitz systems. As we know, the Krylov subspace methods require in each iteration step only products of A with vector and since A is Toeplitz these products can be computed in $\mathcal{O}(n \log n)$ operations via using the fast Fourier transforms (FFTs). However, in order to reduce the number of iterations, iterative method must be chosen with suitable preconditioning in general. The construction of “efficient” preconditioners is the purpose of this paper. Although there exists a rich literature on Hermitian Toeplitz systems (see [3,4] and the references therein), only a few papers consider the non-Hermitian case [5–13]. For Hermitian positive definite Toeplitz matrices, the spectra of the circulant preconditioned Toeplitz matrices are shown to be clustered. It is clear how this affects the convergence of the PCG method. However, for the non-Hermitian case, it is not clear how the clustered eigenvalues affect the convergence of the Krylov subspace methods [3, pp. 74–79]. So searching the efficient preconditioners for non-Hermitian Toeplitz systems is still a promising topic.

[☆] This research is supported by 973 Program (2013CB329404), NSFC (61370147, 61170311, 61170309 and 11301057) and the Fundamental Research Funds for Central Universities.

* Corresponding author.

E-mail addresses: guxianming@live.cn, x.m.gu@rug.nl (X.-M. Gu), tingzhuang@126.com (T.-Z. Huang).

The polynomial preconditioner is a kind of popular and interesting preconditioner in which the main issue is that efficient splitting of the coefficient matrix A is required and a suitable splitting iteration method needs to be used; see, for instance, [14–18]. Suppose that $A = M - N$ represents a splitting of matrix A and $\Omega = M^{-1}N$, where M is nonsingular. If the spectral radius of Ω , denoted by $\rho(\Omega)$, is less than one, i.e., $\rho(\Omega) < 1$, the inverse of A can be written as $A^{-1} = (\sum_{j=0}^{\infty} \Omega^j)M^{-1}$. We take $P_k = M(I + \Omega + \Omega^2 + \cdots + \Omega^{k-1})^{-1}$ as an approximation to matrix A . Then P_k can be used as a preconditioner for linear system (1). We refer to such a preconditioner P_k as a polynomial preconditioner for A . The preconditioned matrix is given by

$$P_k^{-1}A = (I + \Omega + \Omega^2 + \cdots + \Omega^{k-1})M^{-1}A = I - \Omega^k. \quad (2)$$

In preconditioned Krylov subspace methods, the main computational cost is to solve the generalized residual equation $P_k \mathbf{z} = \mathbf{r}$ when P_k is applied as a preconditioner. It follows from (2) that $\mathbf{z} = (I + \Omega + \Omega^2 + \cdots + \Omega^{k-1})M^{-1}\mathbf{r}$. To obtain the vector \mathbf{z} , we perform a k -step iteration as follows:

$$M\mathbf{z}^{(j)} = N\mathbf{z}^{(j-1)} + \mathbf{r}, \quad j = 1, 2, \dots, k. \quad (3)$$

Thus,

$$\mathbf{z}^{(m)} = \Omega^k \mathbf{z}^{(0)} + (I + \Omega + \Omega^2 + \cdots + \Omega^{k-1})M^{-1}\mathbf{r}. \quad (4)$$

If we choose $\mathbf{z}^{(0)} = \mathbf{0}$ in (4), then $\mathbf{z}^{(m)} = (I + \Omega + \Omega^2 + \cdots + \Omega^{k-1})M^{-1}\mathbf{r} = \mathbf{z}$. A polynomial preconditioner is also called a k -step polynomial preconditioner, and it can also be derived from the two-stage iteration methods of trivial outer splittings, refer to [19] for this discussion. Based on the foregoing analysis, the effectiveness of a polynomial preconditioner depends upon a good splitting of matrix A , so that the iteration procedure (3) can be implemented efficiently. The k -step multisplitting preconditioners are commonly applied to parallel and vector machines to solve a parallel linear system [17, 18]. In this paper, by utilizing the efficient CSCS iteration method introduced in [20], we propose a CSCS-based polynomial preconditioner for GMRES method to solve the Toeplitz systems arising in numerical solutions of FDEs.

In Section 2, the CSCS iteration method is briefly reviewed. In Section 3, the polynomial preconditioner based on the CSCS iteration method is established and the spectrum of preconditioned matrix is theoretically analyzed. In Section 4, numerical experiments involving the numerical solutions of FDEs are reported to show the effectiveness of the proposed method. Finally, the paper closes with conclusions in Section 5.

2. The CSCS iteration method

Recently, Ng designed in [20] a circulant and skew-circulant splitting (CSCS) iteration method, which is very efficient for solving the non-Hermitian Toeplitz systems. Based on the fact that the Toeplitz matrix A in (1) naturally possesses circulant/skew-circulant splitting $A = C + S$, where

$$C = \frac{1}{2} \begin{bmatrix} a_0 & a_{-1} + a_{n-1} & \cdots & a_{2-n} + a_2 & a_{1-n} + a_1 \\ a_1 + a_{1-n} & a_0 & \cdots & \cdots & a_{2-n} + a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} + a_2 & \cdots & \cdots & a_0 & a_{-1} + a_{n-1} \\ a_{n-1} + a_{-1} & a_{n-2} + a_{-2} & \cdots & a_1 + a_{1-n} & a_0 \end{bmatrix} \quad (5)$$

and

$$S = \frac{1}{2} \begin{bmatrix} a_0 & a_{-1} - a_{n-1} & \cdots & a_{2-n} - a_2 & a_{1-n} - a_1 \\ a_1 - a_{1-n} & a_0 & \cdots & \cdots & a_{2-n} - a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} - a_2 & \cdots & \cdots & a_0 & a_{-1} - a_{n-1} \\ a_{n-1} - a_{-1} & a_{n-2} - a_{-2} & \cdots & a_1 - a_{1-n} & a_0 \end{bmatrix}. \quad (6)$$

Note that C is a circulant matrix and S is a skew-circulant matrix. A circulant matrix can be diagonalized by the discrete Fourier matrix F and a skew-circulant matrix can also be diagonalized via a discrete Fourier matrix with diagonal scaling, i.e., $\hat{F} = F\Omega$. That is to say, it holds that $C = F^* \Lambda_C F$ and $S = \hat{F}^* \Lambda_S \hat{F}$, where $F = (F)_{j,k} = \frac{1}{\sqrt{n}} e^{\frac{2\pi i}{n}(j-1)(k-1)}$, $\Omega = \text{diag}(1, e^{-\frac{\pi i}{n}}, \dots, e^{-\frac{(n-1)\pi i}{n}})$, $1 \leq j, k \leq n$ and i is the imaginary unit [3, pp. 37–39]. Λ_C and Λ_S are diagonal matrices formed by the eigenvalues of C and S , respectively, which can be obtained in $\mathcal{O}(n \log n)$ operations by using the FFTs. Furthermore, the CSCS iteration method can be algorithmically described as follows:

Let $\mathbf{u}^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $j = 0, 1, 2, \dots$ until the sequence of iterates $\{\mathbf{u}^{(j)}\}_{j=0}^{\infty} \subset \mathbb{C}^n$ converges, compute the next iterate $\mathbf{u}^{(j+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + C)\mathbf{u}^{(j+\frac{1}{2})} = (\alpha I - S)\mathbf{u}^{(j)} + \mathbf{f}, \\ (\alpha I + S)\mathbf{u}^{(j+1)} = (\alpha I - C)\mathbf{u}^{(j+\frac{1}{2})} + \mathbf{f}, \end{cases}$$

where α is a given positive constant and I represents the identity matrix.

After straightforward derivations we can reformulate the CSCS iteration scheme as follows,

$$\mathbf{u}^{(j+1)} = \mathcal{M}(\alpha)\mathbf{u}^{(j)} + \mathcal{G}(\alpha)\mathbf{f}, \quad j = 0, 1, 2, \dots, \quad (7)$$

where

$$\mathcal{M}(\alpha) = (\alpha I + S)^{-1}(\alpha I - C)(\alpha I + C)^{-1}(\alpha I - S) \quad \text{and} \quad \mathcal{G}(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + C)^{-1}. \quad (8)$$

Note that $\mathcal{M}(\alpha)$ is the iteration matrix of the CSCS iteration method. In fact, (7) can also be derived from the following splitting

$$A = M(\alpha) - N(\alpha) \quad (9)$$

of the coefficient matrix A , where

$$M(\alpha) = \frac{1}{2\alpha}(\alpha I + C)(\alpha I + S) \quad \text{and} \quad N(\alpha) = \frac{1}{2\alpha}(\alpha I - C)(\alpha I - S). \quad (10)$$

Concerning the convergence of CSCS iteration method, we have the following theorem established in [20].

Theorem 2.1. Let C and S be the circulant and skew-circulant matrices given in (5) and (6), and α be a positive constant. If C and S are positive definite, then the spectral radius $\rho(\mathcal{M}(\alpha))$ of the CSCS iteration matrix $\mathcal{M}(\alpha) = (\alpha I + S)^{-1}(\alpha I - C)(\alpha I + S)^{-1}(\alpha I - S)$ satisfies $\rho(\mathcal{M}(\alpha)) \leq \sigma(\alpha)$, where

$$\sigma(\alpha) \equiv \max_{\lambda_j \in \lambda(C)} \left| \frac{\alpha - \lambda_j}{\alpha + \lambda_j} \right| \max_{\mu_j \in \lambda(S)} \left| \frac{\alpha - \mu_j}{\alpha + \mu_j} \right|, \quad (11)$$

where $\lambda(C)$ denotes the spectrum of the matrix C . Therefore, it holds that

$$\rho(\mathcal{M}(\alpha)) \leq \sigma(\alpha) < 1 \quad \text{for all } \alpha > 0, \quad (12)$$

i.e., the CSCS iteration converges to the unique solution $\mathbf{x}^* \in \mathbb{C}^n$ of the system (1) for any initial guess.

Theorem 2.1 implies that the CSCS iteration method is convergent for any positive parameter α . However, the optimal value α_* of α is difficult to determine, see [20] and references therein.

3. CSCS-based polynomial preconditioner

Inspired by the pioneer work of [19], we consider to solve the Toeplitz system (1) using Krylov subspace methods such as GMRES with a polynomial preconditioner. The proposed polynomial preconditioner is built by utilizing the CSCS splitting in (9), that is, the splitting of $A = M(\alpha) - N(\alpha)$ of the coefficient matrix A , with the same definitions of $M(\alpha)$ and $N(\alpha)$ as Eqs. (10). Thus,

$$\Omega(\alpha) = M^{-1}(\alpha)N(\alpha) = \mathcal{M}(\alpha) \quad (13)$$

and $\Omega(\alpha)^k = \mathcal{M}(\alpha)^k = [(\alpha I + S)^{-1}(\alpha I - C)(\alpha I + C)^{-1}(\alpha I - S)]^k$. Take

$$P_k(\alpha) = M(\alpha)(I + \Omega(\alpha) + \Omega(\alpha)^2 + \dots + \Omega(\alpha)^{k-1})^{-1}. \quad (14)$$

Then $P_k(\alpha)$ is the induced k -step CSCS preconditioning matrix, concisely denoted as the CSCS(k) preconditioner. In particular, we denote the CSCS(k) preconditioned GMRES method by CSCS(k)-GMRES. The proposed CSCS(k)-preconditioned matrix is $(P_k(\alpha))^{-1}A = I - \Omega(\alpha)^k$. Based on the fact that the spectral radius of $\Omega(\alpha)$ is less than one, as shown in Theorem 2.1, the following conclusion, which describes the spectral distribution of the preconditioned matrix $(P_k(\alpha))^{-1}A$, can be easily obtained.

Proposition 3.1. Let $\Omega(\alpha)$ be the matrix defined in (13) and denoted by $\rho(\Omega(\alpha))$ its spectral radius. Then

- (1) $\rho(\Omega(\alpha)) \leq \sigma(\alpha) < 1$, where $\sigma(\alpha)$ is defined in (11);
- (2) The eigenvalues of $(P_k(\alpha))^{-1}A$ are located in a circle centered at $(1, 0)$ with radius $\rho(\Omega(\alpha)^k)$.

Remark 1. Proposition 3.1 implies that a larger positive integer k in the CSCS(k) preconditioner could lead to a smaller radius of the spectral distribution circle such that the eigenvalues of the CSCS(k)-preconditioned matrix $P_k(\alpha)^{-1}A$ are clustered tightly around 1. However, a larger value of k undoubtedly results in a high computational cost of the CSCS(k)-GMRES method. Hence, the determination of an appropriate positive integer k is the key issue for improving computational efficiency. However, it is difficult to theoretically determine an optimal k when solving a linear system with the CSCS(k)-GMRES method. In the numerical examples, we experimentally discover the suitable k for each example by taking both the computing efficiency and computing time into account.

At the end of this section, it is worth specifying that the preconditioned matrix $(P_k(\alpha))^{-1}A$ does not need to be formed explicitly since $(P_k(\alpha))^{-1}\mathbf{v}$ can be evaluated for any vector \mathbf{v} via a series of matrix–vector products. From Eqs. (8) and (13)–(14), we can find that the core of achieving the $(P_k(\alpha))^{-1}\mathbf{v}$ is to compute the matrix–vector product $\Omega(\alpha)\mathbf{v} = \mathcal{M}(\alpha)\mathbf{v}$, so we give the following algorithm.

Algorithm 1. Calculate the matrix–vector product. $\mathbf{w} = \Omega(\alpha)\mathbf{v} = \mathcal{M}(\alpha)\mathbf{v}$

1. Compute the product $\mathbf{w}_1 = (\alpha I - S)\mathbf{v}$ by using the FFTs;
2. Let $(\alpha I + C)^{-1}\mathbf{w}_1 = \mathbf{w}_2$, i.e., solve the linear systems $(\alpha I + C)\mathbf{w}_2 = \mathbf{w}_1$ via the FFTs;
3. Compute the product $\mathbf{w}_3 = (\alpha I - C)\mathbf{w}_2$ by using the FFTs;
4. Let $(\alpha I + S)^{-1}\mathbf{w}_3 = \mathbf{w}_4$, i.e., solve the linear systems $(\alpha I + S)\mathbf{w}_4 = \mathbf{w}_3$ via the FFTs;
5. Output the result $\mathbf{w} = \mathbf{w}_4$.

Here, it notes that [Algorithm 1](#) can be efficiently implemented by eight FFTs in $\mathcal{O}(n \log n)$ operations [4,20]. Moreover, in actual applications of the CSCS(k) preconditioner $P_k(\alpha)$ (see (14)) to the GMRES method, we need to solve the generalized residual equation $P_k(\alpha)\mathbf{z} = \mathbf{r}$ in each iteration step, where \mathbf{r} is the current residual and \mathbf{z} the generalized residual. For simplicity, we take $k = 3$ and give the following algorithm for solving $P_3(\alpha)\mathbf{z} = \mathbf{r}$. To compute $(P_3(\alpha))^{-1}\mathbf{r} = \mathbf{z}$, we rewrite it as the style of [18]

$$\mathbf{z} = (P_3(\alpha))^{-1}\mathbf{r} = (I + \Omega(\alpha) + \Omega(\alpha)^2)(M(\alpha))^{-1}\mathbf{r} = (I + (I + \Omega(\alpha))\Omega(\alpha))(M(\alpha))^{-1}\mathbf{r}.$$

Algorithm 2. Solve the sub-linear system $P_3(\alpha)\mathbf{z} = \mathbf{r}$.

1. Let $(M(\alpha))^{-1}\mathbf{r} = \mathbf{z}_1$ and solve $M(\alpha)\mathbf{z}_1 = \mathbf{r}$, i.e., the linear system $[\frac{1}{2\alpha}(\alpha I + C)(\alpha I + S)]\mathbf{z}_1 = \mathbf{r}$ can be solved efficiently via four FFTs.
2. Compute $\mathbf{z}_2 = \Omega(\alpha)\mathbf{z}_1$ via [Algorithm 1](#).
3. Compute $\mathbf{z}_3 = \Omega(\alpha)\mathbf{z}_2$ by using [Algorithm 1](#).
4. Output the result $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3$.

From [Algorithms 1–2](#), we can conclude that the total computational complexity of the proposed polynomial preconditioned GMRES method is still in $\mathcal{O}(n \log n)$.

4. Numerical experiments

In this section, two numerical experiments are presented to show the effectiveness of the proposed CSCS(k)-GMRES for solving the non-Hermitian Toeplitz system (1). All experiments are performed in MATLAB R2011b on Intel(R) Pentium(R) CPU G640 @ 2.80 GHz and 4.00 GB of RAM, with machine precision 10^{-16} . In our calculations, the stopping criterion in the GMRES(m) method ($m = 20$) is $\frac{\|\mathbf{r}_q\|_2}{\|\mathbf{r}_0\|_2} < 10^{-8}$, where \mathbf{r}_q is the residual after q th iteration in the GMRES(m) method and the zero vector is the initial guess. For convenience, the right-hand side vectors \mathbf{f} in two following examples are chosen as $\mathbf{f} = A\mathbf{e}$ such that $\mathbf{e} = (1, 1, \dots, 1)^T$ are the exact solutions. The optimal parameters α_* employed in CSCS-based preconditioner have been obtained experimentally. In fact, the experimentally found optimal parameters α_* are the ones resulting in the least number of iterations and CPU time elapsed. Here we consider two real nonsymmetric Toeplitz matrices A arising in the numerical solutions of the following FDEs [1]:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = d_+ \frac{\partial^\gamma u(x, t)}{\partial_+ x^\gamma} + d_- \frac{\partial^\gamma u(x, t)}{\partial_- x^\gamma}, & x \in (0, 2), t \in [0, 1], \\ u(0, t) = u(2, t) = 0, & 0 \leq t \leq 1, \\ u(x, 0) = \exp\left(-\frac{(x - x_c)^2}{2\sigma^2}\right), & x_c = 1.2, \sigma = 0.08. \end{cases} \quad (15)$$

Using the implicit Euler scheme to discrete (15), the resulting system can be written as $A\mathbf{u} := (\frac{\Delta x^\gamma}{\Delta t}I + \tilde{A})\mathbf{u} = \mathbf{f}$ with $\tilde{A} := d_+G_\gamma + d_-G_\gamma^T$, where Δx^γ , Δt , G_γ , \mathbf{u} and \mathbf{f} are similarly defined as those in [1]. Meanwhile, Qu et al. had proved that the CSCS iteration method converges to the exact solution of discretized system arising in fractional advection–diffusion equations (FADEs) for all initial guesses with $\alpha > 0$, see [21] for details. It means that the convergence of the CSCS iteration method for the discretized system of FDEs¹ can be analogously proved by using Qu’s method. Therefore, from a theoretical point of view, the proposed CSCS-based preconditioner and [Proposition 3.1](#) are guaranteed efficiently for handling the resulted Toeplitz linear systems.

In [Tables 1–2](#), we report the number of iterations (“IT”) and CPU consuming time in seconds (“Time”) required for the convergence of different preconditioned GMRES(m) methods for solving the discretized systems of FDEs (15). In the tables, I means no preconditioner is used and C , S are the T. Chan preconditioner and Strang preconditioner, which are two efficient circulant preconditioners designed in [1], respectively. As seen from [Tables 1–2](#), we see that the proposed CSCS-based preconditioners outperform the preconditioners of T. Chan and Strang considerably in aspects of both the number of iterations and CPU time needed for solving different resulted systems. Moreover, it also shows that the accelerating

¹ Here we specify that Qu et al. exploit the 2-order difference scheme to discrete the FADEs and then the authors prove the convergence of CSCS iteration method by using the property of 2-order difference scheme, refer to [21]. In this paper, although we use the 1-order difference scheme to discrete the FDEs and obtain the different property of 1-order difference scheme [1,2], but it is still feasible to prove similarly the convergence of CSCS iteration method by exploiting Qu’s strategies.

Table 1Numerical results of iterative solvers with different preconditioners ($d_+ = 0.9$, $d_- = 0.1$, $\gamma = 1.2$).

Preconditioner		$n = 2^6$ $M = 32$	$n = 2^7$ $M = 74$	$n = 2^8$ $M = 169$	$n = 2^9$ $M = 388$	$n = 2^{10}$ $M = 891$
I	IT	34	33	32	32	31
	Time (s)	0.0168	0.0165	0.0170	0.0176	0.0185
P_1	α_*	0.62	0.60	0.60	0.59	0.60
	IT	9	8	8	8	8
P_2	Time (s)	0.0085	0.0081	0.0082	0.0089	0.0087
	α_*	0.62	0.60	0.60	0.60	0.61
P_3	IT	5	5	5	5	5
	Time (s)	0.0074	0.0072	0.0075	0.0080	0.0084
C	α_*	0.62	0.62	0.060	0.60	0.60
	IT	4	4	3	3	3
S	Time (s)	0.0068	0.0070	0.0067	0.0072	0.0075
	IT	25	24	24	23	23
	Time (s)	0.0150	0.0152	0.0158	0.0154	0.0171
S	IT	25	24	24	23	23
	Time (s)	0.0152	0.0155	0.0160	0.0156	0.0172

Table 2Numerical results of iterative solvers with different preconditioners ($d_+ = 0.8$, $d_- = 0.2$, $\gamma = 1.5$).

Preconditioner		$n = 2^6$ $M = 91$	$n = 2^7$ $M = 256$	$n = 2^8$ $M = 724$	$n = 2^9$ $M = 2048$	$n = 2^{10}$ $M = 5793$
I	IT	28	27	27	26	26
	Time (s)	0.0155	0.0162	0.0171	0.0162	0.0176
P_1	α_*	0.70	0.70	0.65	0.65	0.66
	IT	8	8	7	7	7
P_2	Time (s)	0.0067	0.0074	0.0068	0.0072	0.0078
	α_*	0.62	0.65	0.70	0.70	0.64
P_3	IT	5	5	5	4	4
	Time (s)	0.0065	0.0068	0.0078	0.0070	0.0071
C	α_*	0.65	0.66	0.70	0.65	0.63
	IT	4	4	4	3	3
S	Time (s)	0.0070	0.0073	0.0080	0.0073	0.0074
	IT	14	13	13	13	12
S	Time (s)	0.0101	0.0104	0.0105	0.0102	0.0122
	IT	13	13	13	13	12
	Time (s)	0.0098	0.0103	0.0108	0.0103	0.0121

benefit of the proposed preconditioners can be considerably independent of discretized size. Besides, we also find that the approximation solutions obtained by the proposed preconditioned GMRES(m) methods will be more accurate than those by circulant preconditioned methods in the numerical experiments.

For further illustrating the effectiveness of the proposed preconditioners, the spectral distribution of different preconditioned matrices are drawn in Fig. 1. As see from Fig. 1, it has been found that the spectra of the proposed preconditioned matrices are more clustered than those of circulant preconditioned matrices. Moreover, Proposition 3.1 can be verified by observing Fig. 1.

5. Conclusion

In this study, a k -step CSCS-based polynomial preconditioner is proposed to accelerate GMRES method for solving the non-Hermitian Toeplitz systems arising in numerical solutions of FDEs. Theoretical and experimental results showed the effectiveness of the proposed methods. In future work, we will focus on finding useful methods for determining the optimal values of α and k in the proposed method. We remark that if the 2-order difference scheme used in [21] for fractional derivatives is replaced by the 1-order proposed in this paper, the convergence of the CSCS-based polynomial preconditioners can be proved similarly. Moreover, since the implementation of FFT can be highly parallelized, the CSCS iteration can be benefited from parallel computing; the polynomial preconditioners are originally suitable for parallel computations. It means that the proposed CSCS-based preconditioners can be designed as a parallel version of preconditioners.

Acknowledgments

We acknowledge the referees and editor Prof. Alan Tucker for the many detailed and insightful comments that helped us to improve our manuscript.

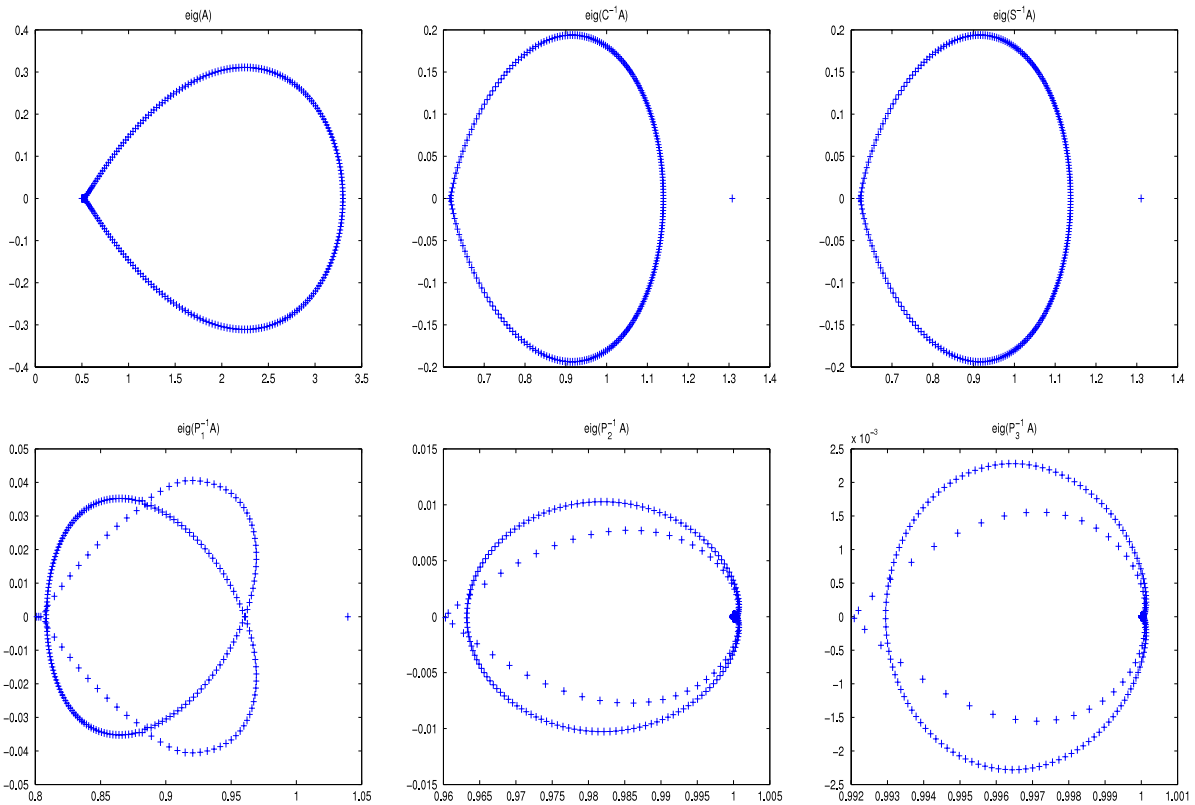


Fig. 1. The spectra of different preconditioned matrices ($n = 2^8$, $M = 724$, $d_+ = 0.8$, $d_- = 0.2$ and $\gamma = 1.5$).

References

- [1] S.-L. Lei, H.-W. Sun, A circulant preconditioner for fractional diffusion equations, *J. Comput. Phys.* 242 (2013) 715–725.
- [2] X.-M. Gu, T.-Z. Huang, X.-L. Zhao, H.-B. Li, L. Li, Strang-type preconditioners for solving fractional diffusion equations by boundary value methods, *J. Comput. Appl. Math.* 277 (2015) 73–86.
- [3] M.K. Ng, *Iterative Methods for Toeplitz Systems*, Oxford University Press, UK, 2004.
- [4] R.H. Chan, M.K. Ng, Conjugate gradient methods for Toeplitz systems, *SIAM Rev.* 38 (1996) 427–482.
- [5] R.H. Chan, W.K. Ching, Toeplitz–circulant preconditioners for Toeplitz systems and their applications to queuing networks with batch arrivals, *SIAM J. Sci. Comput.* 17 (1996) 762–772.
- [6] F. Chen, Y. Jiang, Q. Liu, On structured variants of modified HSS iteration methods for complex Toeplitz linear systems, *J. Comput. Math.* 31 (2013) 57–67.
- [7] D. Potts, G. Steidl, Optimal trigonometric preconditioners for nonsymmetric Toeplitz systems, *Linear Algebra Appl.* 281 (1998) 265–292.
- [8] R.H. Chan, D. Potts, G. Steidl, Preconditioners for non-Hermitian Toeplitz systems, *Numer. Linear Algebra Appl.* 8 (2001) 83–98.
- [9] F.-R. Lin, M.K. Ng, Inverse product Toeplitz preconditioners for non-Hermitian Toeplitz systems, *Numer. Algorithms* 54 (2010) 279–295.
- [10] F. Toutounian, N. Akhoundi, Recursive self preconditioning method based on Schur complement for Toeplitz matrices, *Numer. Algorithms* 62 (2013) 505–525.
- [11] T. Ku, C. Kuo, Spectral properties of preconditioned rational Toeplitz matrices: the non-symmetric case, *SIAM J. Matrix Anal. Appl.* 14 (1993) 521–544.
- [12] R.H. Chan, X.-Q. Jin, Circulant and skew-circulant preconditioners for skew-Hermitian type Toeplitz systems, *BIT* 31 (1991) 632–646.
- [13] T. Huckle, S. Serra-Capizzano, C. Tablino-Possio, Preconditioning strategies for non-Hermitian Toeplitz linear systems, *Numer. Linear Algebra Appl.* 12 (2005) 211–220.
- [14] L.M. Adams, m -step preconditioned conjugate gradient methods, *SIAM J. Sci. Stat. Comput.* 6 (1985) 452–463.
- [15] L.M. Adams, E.G. Ong, Additive polynomial preconditioners for parallel computers, *Parallel Comput.* 9 (1988) 333–345.
- [16] Z.-H. Cao, Y. Wang, On validity of m -step multisplitting preconditioners for linear systems, *Appl. Math. Comput.* 126 (2002) 199–211.
- [17] A. Hadjidimos, A.K. Yeyios, Some notes on multisplitting methods and m -step preconditioners for linear systems, *Linear Algebra Appl.* 248 (1996) 277–301.
- [18] H.-B. Li, T.-Z. Huang, Y. Zhang, X.-P. Liu, On some new approximate factorization methods for block tridiagonal matrices suitable for vector and parallel processors, *Math. Comput. Simul.* 79 (2009) 2135–2147.
- [19] Y.-M. Huang, On m -step Hermitian and skew-Hermitian splitting preconditioning methods, *J. Eng. Math.* (2014) in press. Available online at <http://dx.doi.org/10.1007/s10665-013-9676-z>.
- [20] M.K. Ng, Circulant and skew-circulant splitting methods for Toeplitz systems, *J. Comput. Appl. Math.* 159 (2003) 101–108.
- [21] W. Qu, S.-L. Lei, S.-W. Vong, Circulant and skew-circulant splitting iteration for fractional advection–diffusion equations, *Int. J. Comput. Math.* 91 (2014) 2232–2242.